ROUTH'S THEOREM

R.E. Deakin[1] and M.N. Hunter[2]
[1]Dunsborough, WA, 6281, Australia; [2]Maribyrnong, VIC, 3032, Australia
email: randm.deakin@gmail.com
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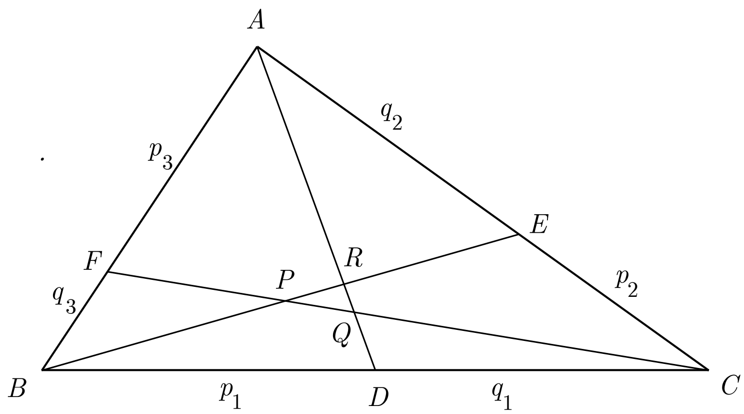

Figure 1.

The sides *AB, BC, CA* of the triangle *ABC* are partitioned by *D, E, F* in the ratios

$$\frac{BD}{DC}=\frac{p_1}{q_1},\quad \frac{CE}{EA}=\frac{p_2}{q_2},\quad \frac{AF}{FB}=\frac{p_3}{q_3} \tag{1}$$

The lines *AD, BE, CF* are known as *cevians*[1] and their intersections *P, Q, R* form a triangle.

Denoting S_{ABC} as the area of triangle *ABC* and S_{PQR} as the area of triangle *PQR* then Routh's theorem gives the ratio of these two areas as

$$\frac{S_{PQR}}{S_{ABC}}=\frac{\left(p_1p_2p_3-q_1q_2q_3\right)^2}{\left(p_1p_2+q_1q_2+p_1q_2\right)\left(p_2p_3+q_2q_3+p_2q_3\right)\left(p_3p_1+q_3q_1+p_3q_1\right)} \tag{2}$$

This theorem of geometry is attributed to Edward John Routh (1831–1907) who gave this in a slightly different form in his 1891 book titled *A Treatise on Analytic Statics with numerous examples* (Routh 1891, p.89). Routh was a fellow of Peterhouse, the oldest college of Cambridge University, England and the most famous of the Cambridge coaches for the Mathematical Tripos[2]. An earlier statement of this theorem (with a proof) was published in *Solutions of the Cambridge Senate-House Problems and Riders for the year 1878,* edited by J.W.L. Glaisher (Glaisher et al., 1879, rider (vii), p. 33-34). James Whitbread Lee Glaisher (1848–1928) was a fellow of Trinity College, Cambridge and acknowledged as the provider of the solution (2) to the problem described above.

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[1] A cevian is a line that intersects both a triangle's vertex and the side opposite. The name 'cevian' comes from the Italian mathematician Giovanni Ceva who proved a well-known theorem involving cevians.

[2] The Mathematical Tripos is the mathematics course taught in the Faculty of Mathematics at Cambridge University and a Tripos is any examination undertaken by an undergraduate to qualify for a bachelor's degree. Prior to 1824 the Tripos was known as The Senate-House Examination. From the late 1700's to the early 1900's the Tripos consisted of multiple examinations over a fortnight and the best performed student was the senior wrangler, the second-best student was the second wrangler, and so on. (Potts was twenty-sixth wrangler in 1827, Routh was senior wrangler in 1854 and Glaisher was second wrangler in 1871).

An even earlier reference to this subdivision of the triangle is given in 1859 by Robert Potts, (a graduate of Trinity College in 1827), in his School Edition of *Euclid's Elements of Geometry, the first six books* (Potts 1859, problem 100, p. 80)

There are many proofs of Routh's Theorem, for example Coxeter (1969, p.211 and pp. 219-20), Niven (1976), Klamkin and Liu (1981), and Kline and Velleman (1995). The proof below follows Glaisher et al. (1879, rider (vii), pp. 33-34)

A Proof of Routh's Theorem

Consider Figure 2 that is the triangle CAD of Figure 1 and the transversal BE (extended).

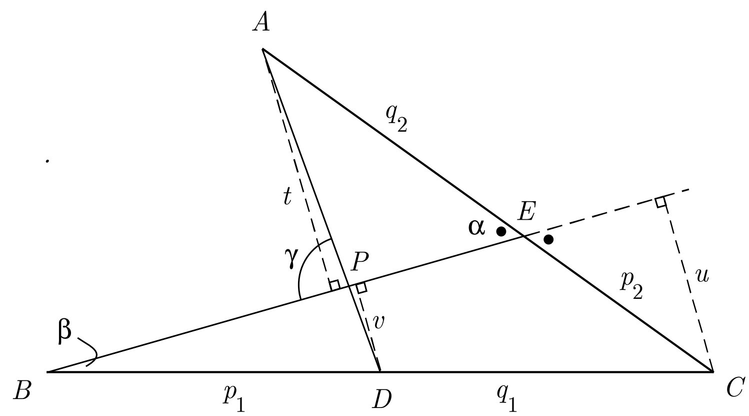


Figure 2.

From the transversal BE construct perpendiculars t, u, v to A, C, and D, and then

$$\sin\alpha = \frac{t}{AE} = \frac{u}{EC} \quad \text{and} \quad \frac{t}{u} = \frac{AE}{EC}$$
$$\sin\beta = \frac{u}{BC} = \frac{v}{BD} \quad \text{and} \quad \frac{u}{v} = \frac{BC}{BD}$$
$$\sin\gamma = \frac{v}{PD} = \frac{t}{AP} \quad \text{and} \quad \frac{v}{t} = \frac{PD}{AP}$$

Multiplying the ratios gives

$$1 = \frac{t}{u} \times \frac{u}{v} \times \frac{v}{t} = \frac{AE}{EC} \times \frac{BC}{BD} \times \frac{PD}{AP}$$

and so

$$\frac{AP}{PD} = \frac{AE}{EC} \times \frac{BC}{BD} = \frac{q_2}{p_2} \times \frac{p_1 + q_1}{p_1} \tag{3}$$

Now consider Figure 3 that shows triangle ABD partitioned by the line BP on the left and triangle ABC partitioned by the line AD on the right.

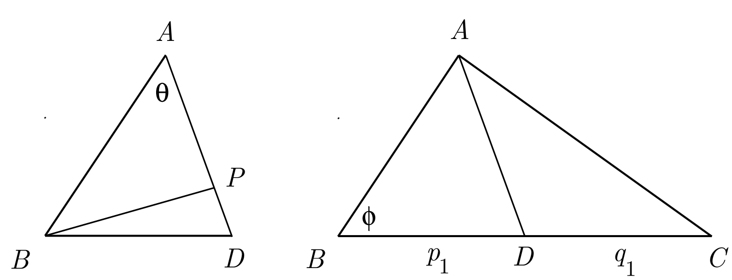


Figure 3

In Figure 3 (Left) the area ratio of the two triangles ABP and ABD is

$$\frac{S_{ABP}}{S_{ABD}} = \frac{\frac{1}{2}(AB)(AP)\sin\theta}{\frac{1}{2}(AB)(AD)\sin\theta} = \frac{AP}{AD} = \frac{AP}{AP + PD} = \frac{1}{1 + \dfrac{PD}{AP}} \tag{4}$$

and using (3) in (4) and simplifying gives

$$\frac{S_{ABP}}{S_{ABD}} = \frac{q_2\left(p_1 + q_1\right)}{p_1 p_2 + q_1 q_2 + p_1 q_2} \tag{5}$$

In Figure 3 (Right) the area ratio of triangles ABC and ABD is

$$\frac{S_{ABD}}{S_{ABC}} = \frac{\frac{1}{2}(BA)(BD)\sin\phi}{\frac{1}{2}(BA)(BC)\sin\phi} = \frac{BD}{BC} = \frac{p_1}{p_1 + q_1} \tag{6}$$

Multiplying the two area ratios (5) and (6) gives

$$\frac{S_{ABP}}{S_{ABC}} = \frac{p_1 q_2}{p_1 p_2 + q_1 q_2 + p_1 q_2} \tag{7}$$

In a similar manner, beginning first with triangle ABE and the transversal FC and then with triangle BCF and the transversal AD the following area ratios can be obtained

$$\frac{S_{CBQ}}{S_{ABC}} = \frac{p_2 q_3}{p_2 p_3 + q_2 q_3 + p_2 q_3} \tag{8}$$

and

$$\frac{S_{ACR}}{S_{ABC}} = \frac{p_3 q_1}{p_3 p_1 + q_3 q_1 + p_3 q_1} \tag{9}$$

Now the area of triangle PQR in Figure 1 can be expressed as

$$S_{PQR} = S_{ABC} - S_{ABP} - S_{CBQ} - S_{ACR}$$

and division by S_{ABC} gives the area ratio

$$\frac{S_{PQR}}{S_{ABC}} = 1 - \frac{S_{ABP}}{S_{ABC}} - \frac{S_{CBQ}}{S_{ABC}} - \frac{S_{ACR}}{S_{ABC}} \tag{10}$$

Substituting the three area ratios (7), (8) and (9) into (10) gives

$$\frac{S_{PQR}}{S_{ABC}} = 1 - \frac{p_1 q_2}{p_1 p_2 + q_1 q_2 + p_1 q_2} - \frac{p_2 q_3}{p_2 p_3 + q_2 q_3 + p_2 q_3} - \frac{p_3 q_1}{p_3 p_1 + q_3 q_1 + p_3 q_1}$$

and after some reduction we have Routh's theorem (2)

$$\frac{S_{PQR}}{S_{ABC}} = \frac{\left(p_1 p_2 p_3 - q_1 q_2 q_3\right)^2}{\left(p_1 p_2 + q_1 q_2 + p_1 q_2\right)\left(p_2 p_3 + q_2 q_3 + p_2 q_3\right)\left(p_3 p_1 + q_3 q_1 + p_3 q_1\right)}$$

Alternative forms of Routh's Theorem

[A]

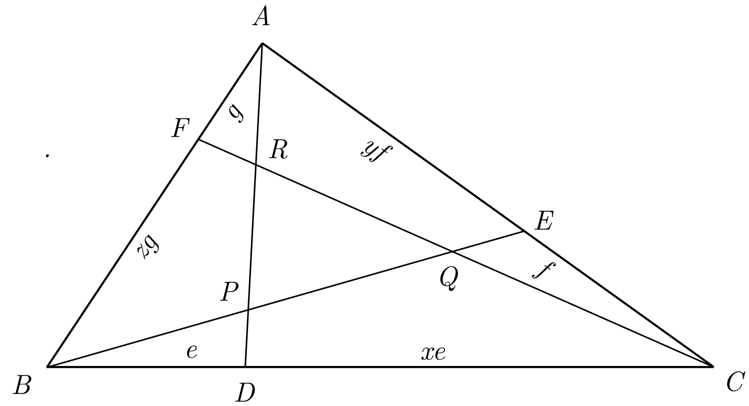


Figure 4

The sides AB, BC, CA of the triangle ABC are partitioned by D, E, F in the ratios

$$\frac{BD}{DC}=\frac{e}{xe}=\frac{1}{x},\quad \frac{CE}{EA}=\frac{f}{yf}=\frac{1}{y},\quad \frac{AF}{FB}=\frac{g}{zg}=\frac{1}{z}\quad \text{where } x,y,z>0 \tag{11}$$

Comparing the ratios (1) with the ratios (11)

$$p_1=p_2=p_3=1,\ q_1=x,\ q_2=y,\ q_3=z$$

and using these in (2) gives Routh's theorem as

$$\frac{S_{PQR}}{S_{ABC}}=\frac{\left(1-xyz\right)^2}{\left(1+x+xz\right)\left(1+y+yx\right)\left(1+z+zy\right)} \tag{12}$$

[B]

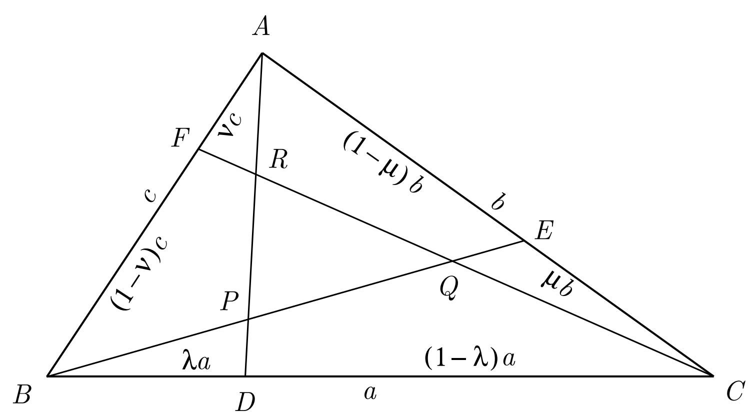


Figure 5

The sides AB, BC, CA of the triangle ABC are partitioned by D, E, F in the ratios

$$\frac{BD}{DC}=\frac{\lambda a}{\left(1-\lambda\right)a}=\frac{\lambda}{1-\lambda},\quad \frac{CE}{EA}=\frac{\mu b}{\left(1-\mu\right)b}=\frac{\mu}{1-\mu},\quad \frac{AF}{FB}=\frac{\nu c}{\left(1-\nu\right)c}=\frac{\nu}{1-\nu}\quad \text{where } \lambda,\mu,\nu>0 \tag{13}$$

Comparing the ratios (1) with the ratios (13)

$$p_1 = \lambda, \quad p_2 = \mu, \quad p_3 = \nu, \; q_1 = 1-\lambda, \; q_2 = 1-\mu, \; q_3 = 1-\nu$$

and using these in (2) gives Routh's theorem as

$$\frac{S_{PQR}}{S_{ABC}} = \frac{\left(\lambda\mu\nu - (1-\lambda)(1-\mu)(1-\nu)\right)^2}{(1-\lambda+\nu\lambda)(1-\mu+\lambda\mu)(1-\nu+\mu\nu)} \tag{14}$$

A Special Result of Routh's Theorem

If, in any triangle *ABC,* the sides *AB, BC, CA* are partitioned by *D, E, F* in the common ratio

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = \frac{1}{2}$$

then Routh's theorem gives

$$\frac{S_{PQR}}{S_{ABC}} = \frac{1}{7}$$

This can be verified by: (i) letting $p_1 = p_2 = p_3 = 1$ and $q_1 = q_2 = q_3 = 2$ in (2);
(ii) letting $x = y = z = 2$ in (12); and
(iii) letting $\lambda = \mu = \nu = \frac{1}{3}$ in (14).

The triangle *ABC* partitioned in this way is sometimes called *Feynman's Triangle* and arises from the problem posed to the American physicist Richard Feynman

> For a triangle in the plane, if each vertex is joined to the point one-third along the opposite side (measured say anti-clockwise), prove that the area of the inner triangle formed by these lines is exactly one-seventh of the area of the initial triangle. (Durán-Camejo 2010)

It is said that this problem was posed by Kai Li Chung, from Stanford University, to Feynman during a dinner conversation after a colloquium at Cornell university. According to Cook and Wood (2004) "Feynman could not believe that the ratio of the areas of the triangles was 1/7 since it had nothing to do with the number three. He spent most of the night trying to disprove it, but finally proved it in the special case when the triangle was equilateral." Cook & Wood (2004) and de Villiers (2005) have several proofs of this special result.

Feynman's proof

Here we follow the proof given in Cook and Wood (2004) for an equilateral triangle ABC where the sides AB, BC, CA are partitioned in the common ratio

$$\frac{BD}{DC}=\frac{CE}{EA}=\frac{AF}{FB}=\frac{1}{2}$$

That is, each vertex is joined to the point one-third along the opposite side, measured anti-clockwise as shown in Figure 6.

Let $AB=BC=CA=3$, thus $AF=BD=CE=1$, $FB=DC=EA=2$ and denoting the area of ABC as S_{ABC} then $S_{ABC}=\frac{1}{2}(AB)(AC)\sin(60^\circ)=\frac{9\sqrt{3}}{4}$.

In the triangle AFC, the side FC can be obtained from the cosine rule where $(FC)^2=(AF)^2+(AC)^2-2(AF)(AC)\cos(60^\circ)$ giving $FC=\sqrt{7}$.

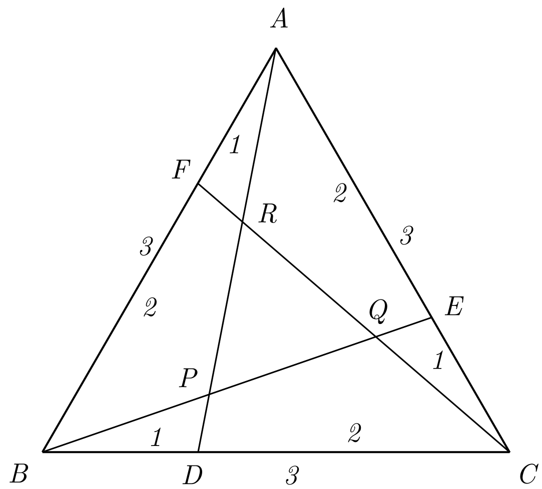


Figure 6

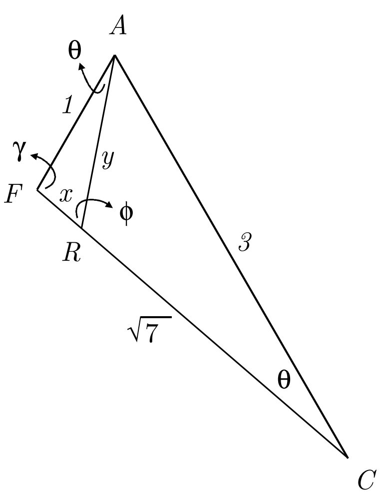


Figure 7

Now consider the similar triangles AFC and AFR shown in Figure 7 noting $\phi=F\hat{R}A=F\hat{A}C=60^\circ$. First, in triangle AFC the sine rule gives $\frac{AF}{\sin\theta}=\frac{FC}{\sin\phi}$ and then $\frac{\sin\theta}{\sin\phi}=\frac{AF}{FC}=\frac{1}{\sqrt{7}}$. Then, in triangle AFR the sine rule gives $\frac{AF}{\sin\phi}=\frac{FR}{\sin\theta}$ and then $\frac{\sin\theta}{\sin\phi}=\frac{FR}{AF}=\frac{x}{1}$.

Equating these ratios gives $x=\frac{1}{\sqrt{7}}$.

Similarly, in triangle AFC, $\frac{\sin\theta}{\sin\gamma}=\frac{AF}{AC}=\frac{1}{3}$ and in triangle AFR, $\frac{\sin\theta}{\sin\gamma}=\frac{FR}{AR}=\frac{x}{y}$ and equating these ratios gives $y=3x=\frac{3}{\sqrt{7}}$.

Now, considering the symmetry of Figure 6, $x=FR=EQ=DP$, $y=AR=CQ=BP$, and the sides of the internal equilateral triangle PQR are $\sqrt{7}-x-y=\frac{3}{\sqrt{7}}$ and the area of the internal triangle PQR is $S_{PQR}=\frac{9\sqrt{3}}{28}$ which is $\frac{1}{7}S_{ABC}$.

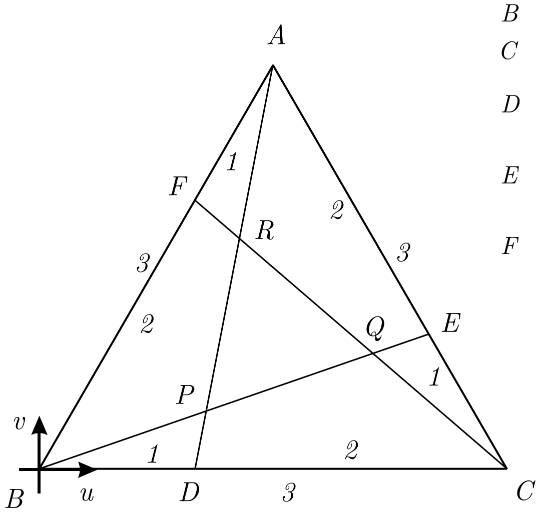
The proof shown here for an equilateral triangle can be generalised and extended to any triangle by understanding that every triangle is an *affine* transformation of an equilateral triangle and that an affine transformation preserves length and area ratios.

An affine transformation of points $i=1,2,\ldots,n$ in a u-v coordinate system to an x-y system can be defined as

$$\begin{aligned} x_i &= au_i+bv_i+c \\ y_i &= du_i+ev_i+f \end{aligned} \tag{15}$$

where a,b,c,d,e and f are constants. c and f are translations between the origins of the u-v and x-y coordinate systems and scale factors and rotation are functions of a, b, d and e.

Suppose that the equilateral triangle ABC and the points DEF of Figure 6 are transformed by (15) with $a = 1,\ b = \dfrac{1}{\sqrt{3}},\ c = 0,\ d = \dfrac{1}{3},\ e = \dfrac{2}{\sqrt{3}},\ f = 0$ to the triangle $A'B'C'$ with points $D'E'F'$. The result is shown below in Figure 8 where the diagram on the left is the original figure, the diagram on the right is the transformed figure and the coordinates u,v (original) and x,y (transformed) are shown in between.



	(u,v)		(x,y)
A	$\left(\frac{3}{2},\frac{3\sqrt{3}}{2}\right)$	A'	$\left(3,\frac{7}{2}\right)$
B	$(0,0)$	B'	$(0,0)$
C	$(3,0)$	C'	$(3,1)$
D	$(1,0)$	D'	$\left(1,\frac{1}{3}\right)$
E	$\left(\frac{5}{2},\frac{\sqrt{3}}{2}\right)$	E'	$\left(3,\frac{11}{6}\right)$
F	$\left(1,\sqrt{3}\right)$	F'	$\left(2,\frac{7}{3}\right)$

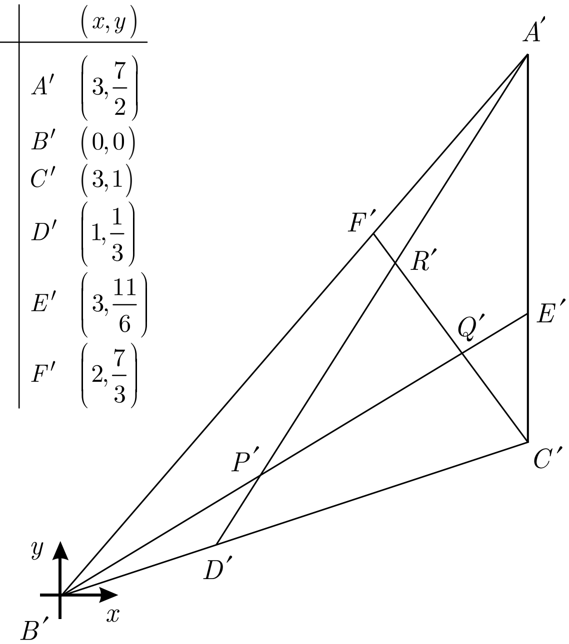


Figure 8

From the coordinate data in Figure 8 the following length ratios can be obtained:

$$A'B' = \frac{\sqrt{85}}{2},\ A'F' = \frac{\sqrt{85}}{6} \quad \text{and} \quad \frac{A'F'}{A'B'} = \frac{1}{3}$$
$$B'C' = \sqrt{10},\quad B'D' = \frac{\sqrt{10}}{3} \quad \text{and} \quad \frac{B'D'}{B'C'} = \frac{1}{3}$$
$$C'A' = \frac{5}{2},\qquad C'E' = \frac{5}{6} \quad \text{and} \quad \frac{C'E'}{C'A'} = \frac{1}{3}$$

and these demonstrate that the affine transformation preserves length ratios in this case.

Also, from the coordinate data and using the rule that the area of a triangle is equal to $\frac{1}{2}\left(\text{base} \times \text{height}\right)$, then denoting areas of triangles as S_{XYZ} that following area ratios can be obtained

$$S_{ABC} = \frac{9\sqrt{3}}{4},\quad S_{ABD} = \frac{3\sqrt{3}}{4} \quad \text{and} \quad \frac{S_{ABD}}{S_{ABC}} = \frac{1}{3}$$
$$S_{A'B'C'} = \frac{15}{4},\quad S_{A'B'D'} = \frac{5}{4} \quad \text{and} \quad \frac{S_{A'B'D'}}{S_{A'B'C'}} = \frac{1}{3}$$

and these area ratios demonstrate that the affine transformation preserves area ratios in this case.

A Special Case of Routh's Theorem for the Equilateral triangle ABC

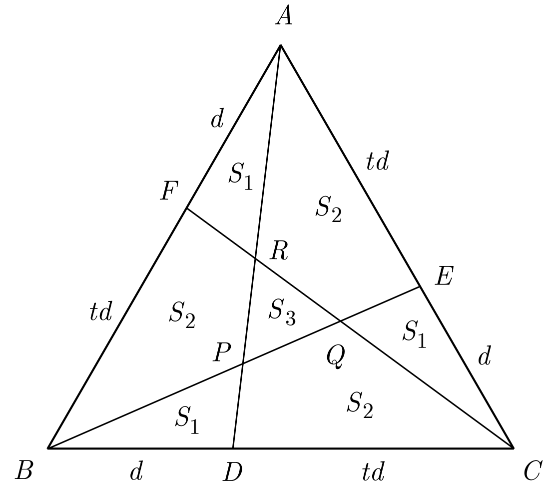


Figure 9. ABC is an equilateral triangle and S_1, S_2, S_3 are areas

The sides AB, BC, CA of the equilateral triangle ABC are partitioned by D, E, F in the common ratio

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = \frac{d}{td} = \frac{1}{t} \quad \text{where } t > 0 \tag{16}$$

Routh's theorem for this special case is

$$\frac{S_{PQR}}{S_{ABC}} = \frac{S_3}{3\left(S_1 + S_2\right) + S_3} = \frac{\left(t^3 - 1\right)^2}{\left(1 + t + t^2\right)^3} = \frac{\left(t - 1\right)^2\left(1 + t + t^2\right)^2}{\left(1 + t + t^2\right)^3} = \frac{\left(t - 1\right)^2}{1 + t + t^2} \tag{17}$$

where S_1, S_2, S_3 denote areas shown in Figure 9.

Also, comparing ratios (1) with ratios (16)

$$p_1 = p_2 = p_3 = 1, \quad q_1 = q_2 = q_3 = t$$

and using these in (5) and (7) gives

$$\frac{S_{ABP}}{S_{ABD}} = \frac{S_{ACQ}}{S_{BCE}} = \frac{S_{ACR}}{S_{ACF}} = \frac{S_1 + S_2}{2S_1 + S_2} = \frac{t\left(1 + t\right)}{1 + t + t^2} \tag{18}$$

and

$$\frac{S_{ABP}}{S_{ABC}} = \frac{S_{BCQ}}{S_{ABC}} = \frac{S_{ACR}}{S_{ABC}} = \frac{S_1 + S_2}{3\left(S_1 + S_2\right) + S_3} = \frac{t}{1 + t + t^2} \tag{19}$$

Dividing (17) by (19) gives the ratio

$$\frac{S_3}{S_1 + S_2} = \frac{\left(t - 1\right)^2}{t} \tag{20}$$

Rearranging (18) and simplifying gives a quadratic equation in t as

$$S_1 t^2 + S_1 t - \left(S_1 + S_2\right) = 0$$

with the solution

$$t = \frac{-S_1 \pm \sqrt{S_1^2 + 4S_1\left(S_1 + S_2\right)}}{2S_1} = \frac{-S_1 \pm \sqrt{S_1\left(5S_1 + 4S_2\right)}}{2S_1} \tag{21}$$

Special Results for ratios $\frac{S_3}{S_{ABC}}$, $\frac{S_1 + S_2}{S_{ABC}}$ and $\frac{S_3}{S_1 + S_2}$ in the Equilateral Triangle of Figure 9

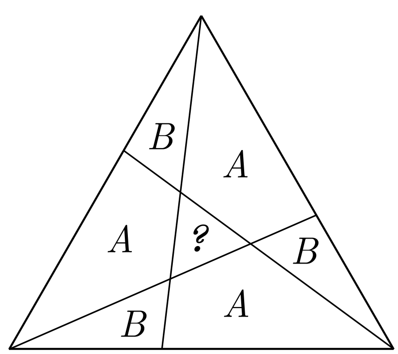
t	$\frac{S_3}{S_{ABC}} = \frac{(t-1)^2}{1+t+t^2}$	$\frac{S_1+S_2}{S_{ABC}} = \frac{t}{1+t+t^2}$	$\frac{S_3}{S_1+S_2} = \frac{(t-1)^2}{t}$
2	1/7	2/7	1/2
3	4/13	3/13	4/3
4	9/21	4/21	9/4
5	16/31	5/31	16/5
3/2	1/19	6/19	1/6
5/2	3/13	10/39	9/10

Table 1

Curly's Conundrum No. 20

Curly's Conundrum No. 20

The equilateral triangle shown in the figure below is divided by three straight lines. The area of each quadrilateral "A" is 22 square units and the area of each triangle "B" is 8 square units. What is the area of the centre triangle?



The Institution of Surveyors Victoria (ISV) has a news bulletin *Traverse* that is published quarterly and circulated to members. In *Traverse 120* (Nov., 1991) the puzzle (above) was published as *Curly's Conundrum No. 20.* In the next issue, rather than the solution, the following note appeared:

> (Well, Curly has finally tripped himself up!)
> I am embarrassed! A solution to Conundrum No. 20 is beyond me. The puzzle was published in The Australian *Mind of the Year* Contest for 1989 and was one of the questions posed to the five finalists.
> The answer, published the following week was five (5) square units.
> **PS:** Please send me a solution as this problem has been driving me crazy for years.

Curly's Conundrum No. 20 has been re-published in *Traverse 161, 202, 243, 284* and *325*. And in each subsequent issue the same mea culpa as above has appeared. The two solutions below may be helpful

Solution to Curly's Conundrum No. 20 Using Routh's Theorem

1 The diagram and given information suggests that the special case of Routh's theorem is applicable and that areas $S_1 = B = 8$ and $S_2 = A = 22$ (comparing Curly's diagram with Figure 9).

The factor t can be obtained from (21) as

$$t = \frac{-S_1 \pm \sqrt{S_1\left(5S_1 + 4S_2\right)}}{2S_1} = \frac{-8 \pm 32}{16} = \frac{3}{2} \quad \text{(taking positive value)}$$

2 S_3, the area of the small triangle in the middle of the diagram, can be obtained from the ratio (20) as

$$S_3 = \left(S_1 + S_2\right)\frac{\left(t-1\right)^2}{t} = \left(30\right)\frac{\left(\frac{1}{2}\right)^2}{\frac{3}{2}} = \left(30\right)\frac{1}{6} = 5$$

The total area of the figure is $3A + 3B + S_3 = 95$ square units and the ratio of the area of the small inner triangle to the total area is $5/95 = 1/19$.

For an equilateral triangle of area $S = 95\ \text{m}^2$ having sides of length l then $l = \sqrt{\frac{4S}{\sqrt{3}}} = 14.811924$ m. If $l = td + d = d\left(1+t\right)$ [see Figure 9] and $t = 3/2$ then $d = 5.924770$ m and $td = 8.887154$ m.

The dimensions of the figure in *Curly's Conundrum No. 20* (lengths in metres) are

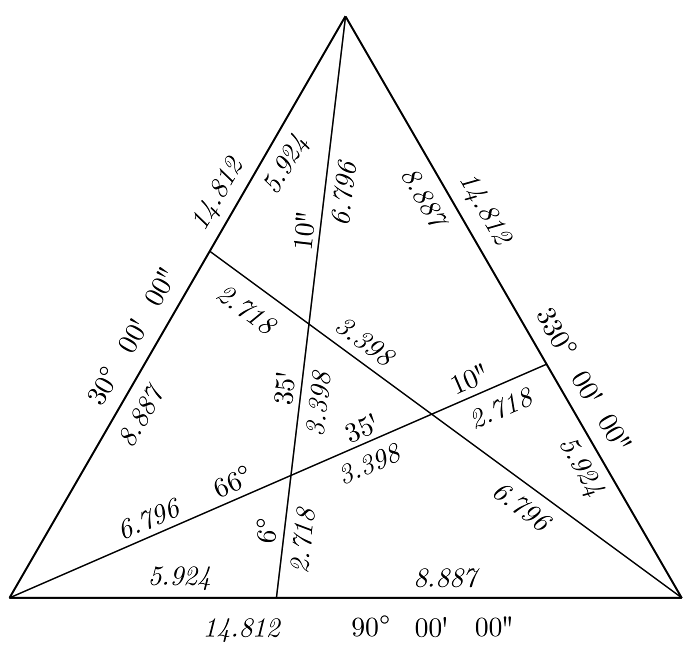


Figure 10

Solution to Curly's Conundrum No. 20 Not using Routh's Theorem

1 Let Figure 9 describe Curly's problem where areas $S_1 = B = 8$, $S_2 = A = 22$ and the area of triangle *PQR,* denoted as $S_{PQR} = S_3$ is required. Now, let $d = AF = BD = CE$, $td = FB = DC = EA$ and the sides of triangle ABC are $AB = BC = CA = d\left(1+t\right)$.

Using the cosine rule in the triangle ABD gives $\left(AD\right)^2 = \left(BD\right)^2 + \left(AB\right)^2 - 2\left(BD\right)\left(AB\right)\cos\left(60^\circ\right)$ and

$$AD = AP + PD = d\sqrt{1+t+t^2} \tag{22}$$

2 In Figure 9, the triangles ABP and ABD (having areas S_{ABP}, S_{ABD} respectively) have a common side AB and a common angle A and using the formula for the area of a plane triangle the area ratio

$$\frac{S_{ABP}}{S_{ABD}} = \frac{\frac{1}{2}\left(AB\right)\left(AP\right)\sin A}{\frac{1}{2}\left(AB\right)\left(AD\right)\sin A} = \frac{AP}{AD} = \frac{AP}{AP+PD} = \frac{1}{1+\dfrac{PD}{AP}} \tag{23}$$

where $S_{ABP} = S_1 + S_2$ and $S_{ABD} = 2S_1 + S_2$

The area ratio on the left-hand-side of (23) is a known quantity and the right-hand-side has an unknown length ratio PD/AP. An expression for this ratio can be obtained as follows

3 Figure 11 shows similar triangles ABD and BPD. Using the sine rule in triangle ABD gives $\dfrac{d}{\sin\theta} = \dfrac{AD}{\sin\phi} = \dfrac{d\left(1+t\right)}{\sin\gamma}$ and ratios

$$\frac{\sin\theta}{\sin\phi} = \frac{d}{AD},\quad \frac{\sin\theta}{\sin\gamma} = \frac{1}{1+t},\quad \frac{\sin\phi}{\sin\gamma} = \frac{AD}{d\left(1+t\right)} \tag{24}$$

Using the sine rule in triangle BPD gives $\dfrac{x}{\sin\theta} = \dfrac{d}{\sin\phi} = \dfrac{y}{\sin\gamma}$ and ratios

$$\frac{\sin\theta}{\sin\phi} = \frac{x}{d},\quad \frac{\sin\theta}{\sin\gamma} = \frac{x}{y},\quad \frac{\sin\phi}{\sin\gamma} = \frac{d}{y} \tag{25}$$

Equating ratios in (24) and (25), and using (22) gives

$$\begin{aligned} \frac{\sin\theta}{\sin\phi} &= \frac{d}{AD} = \frac{x}{d} \quad \text{giving} \quad x = \frac{d^2}{AD} = \frac{d}{\sqrt{1+t+t^2}} \\ \frac{\sin\theta}{\sin\gamma} &= \frac{1}{1+t} = \frac{x}{y} \quad \text{giving} \quad y = x\left(1+t\right) = \frac{d\left(1+t\right)}{\sqrt{1+t+t^2}} \end{aligned} \tag{26}$$

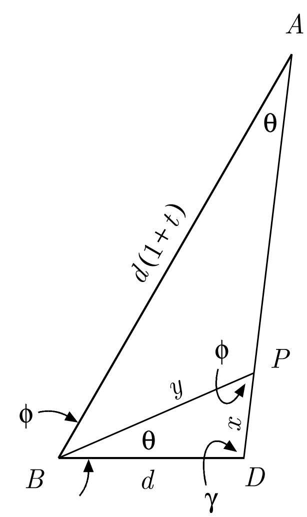


Figure 11

Now $AP = AD - x = d\sqrt{1+t+t^2} - \dfrac{d}{\sqrt{1+t+t^2}} = \dfrac{dt\left(1+t\right)}{\sqrt{1+t+t^2}}$ and $PD = x = \dfrac{d}{\sqrt{1+t+t^2}}$ so the ratio PD/AP

$$\frac{PD}{AP} = \frac{1}{t\left(1+t\right)} \tag{27}$$

4 Substituting (27) in (23) and simplifying gives

$$\frac{S_{ABP}}{S_{ABD}} = \frac{2S_1 + S_2}{S_1 + S_2} = \frac{t\left(1+t\right)}{1+t+t^2}$$

which can be re-arranged into a quadratic equation in t as

$$S_1 t^2 + S_1 t - \left(S_1 + S_2\right) = 0$$

with the solution

$$t = \frac{-S_1 \pm \sqrt{S_1^2 + 4S_1\left(S_1 + S_2\right)}}{2S_1} = \frac{-S_1 \pm \sqrt{S_1\left(5S_1 + 4S_2\right)}}{2S_1} \tag{28}$$

Substituting $S_1 = 8$ and $S_2 = 22$ into (28) and solving for the positive value of t gives

$$t = \frac{-8 + \sqrt{1024}}{16} = \frac{3}{2} \tag{29}$$

5 By symmetry and with reference to Figures 11 and 9, $AR = BP = y$ and $PR = AD - x - y$ and using (26) and (22) gives

$$PR = \frac{d\left(t^2 - 1\right)}{\sqrt{1+t+t^2}}$$

and since $PR = RQ = PQ$ the area of the equilateral triangle PQR is

$$S_{PQR} = \tfrac{1}{2}\left(PR\right)\left(PQ\right)\sin P = \frac{d^2\left(t^2-1\right)^2}{1+t+t^2}\left(\frac{\sqrt{3}}{4}\right) \tag{30}$$

6 An expression for d^2 can be obtained from the area of triangle ABD as

$$S_{ABD} = \tfrac{1}{2}\left(AB\right)\left(BD\right)\sin\left(60^\circ\right) = d^2\left(1+t\right)\frac{\sqrt{3}}{4} = 2S_1 + S_2$$

substituting $S_1 = 8$ and $S_2 = 22$ gives

$$d^2 = \frac{4}{\sqrt{3}}\left(\frac{38}{1+t}\right) \tag{31}$$

7 Substituting (31) into (30) and using (29) gives

$$S_{PQR} = \frac{38}{1+t}\left(\frac{\left(t^2-1\right)^2}{1+t+t^2}\right) = \frac{38\left(\frac{5}{4}\right)^2}{\left(\frac{5}{2}\right)\left(\frac{19}{4}\right)} = 5$$

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